ON THE VAN KAMPEN THEOREM

M. ARTIN† and B. MAZUR‡

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§1. THE MAIN THEOREM

Given an open covering \( \{U_i\} \) of a topological space \( X \), there is a spectral sequence relating the homology of the intersections of the \( U_i \) to the homology of \( X \). The van Kampen theorem [4, 5] describes \( \pi_1(X) \) in terms of the fundamental groups of the \( U_i \) and their intersections, and the object of this paper is to provide a generalization of this result, analogous to the spectral sequence for homology, to the higher homotopy groups.

We work in the category of reduced simplicial sets (the reduced semi-simplicial complexes of Kan [5]). Let \( f: U \rightarrow X \) be a surjective morphism in this category, and let \( U^l \) denote the \( l \)-fold fibre product of \( U \) with itself over \( X \). Then \( U^l \) is the term of dimension \( l - 1 \) of a simplicial object \( U^* \) in the category of simplicial sets in the well-known way. Therefore for varying \( l \) the \( q \)-th homotopy groups \( \pi_q(U^l) \) form a simplicial group \( \pi_q(U^*) \), and hence groups \( \pi_p(\pi_q(U^*)) \) are defined for \( p \geq 0, q \geq 1 \). These groups are abelian for \( (p, q) \neq (0, 1) \).

Our result is the following:

**Theorem.** There is a spectral sequence of homological type whose term \( E^{2}_{p, q} \) is \( \pi_p(\pi_q(U^*)) \) and whose abutment is the associated graded group of a certain filtration of \( \pi_q(X) \), i.e.

\[
E^{2}_{p, q} = \pi_p(\pi_q(U^*)) \Rightarrow \pi_{p+q}(X).
\]

For notational convenience, we define \( E^{1}_{p, q} = \pi_q(U^{p+1}) \), so that \( E^{1}_{*, q} \) is a simplicial group whose homotopy is \( E^{2}_{*, q} \).

Consider the case that \( U \) is the wedge of a family of simplicial subsets \( U_j \) of \( X (j \in J) \), which cover \( X \). We call such a map \( f \) a covering. The fibre product \( U^{l+1} \) is then the wedge of intersections of the \( U_j \):

\[
U^{l+1} = \bigvee_{(j) \in J^{l+1}} U_{(j)}
\]

where \( (j) = (j_0, \ldots, j_l) \) and \( U_{(j)} = U_{j_0} \cap \cdots \cap U_{j_l} \). Thus for coverings, the \( E^2 \)-term of the spectral sequence may be computed from the homotopy groups of wedges of intersections of the \( U_j \).

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If the index set $J$ is well-ordered, the covering $f$ is called ordered, and we obtain a spectral sequence of an ordered covering having smaller $E^2$-term:

$$E_{p,q}^2 = \pi_p\left(\bigvee_{(j)} U_{(j)}\right) \Rightarrow \pi_{p+q}(X),$$

where the wedge is taken only over those $(j)$ such that $j_0 \leq j_1 \leq \cdots \leq j_l$. It seems clear that there is a relationship between the triad homotopy theory of Blakers and Massey [1] and this spectral sequence.

Note that in the analogous spectral sequence for homology

$$E_{p,q}^2 = H_p\left(\bigvee_{(j)} U_{(j)}\right) \Rightarrow H_{p+q}(X)$$

it suffices to take the wedge over those $(j)$ for which $j_0 < \cdots < j_l$. But for the homotopy spectral sequence the degenerate intersections are essential—excluding them, $\pi_q(\bigvee U_{(j)})$ would even lack the degeneracy operators needed to make it a simplicial group.

Suppose that $A$ and $B$ are simply connected. Then since the homotopy of a wedge $A \vee B$ splits canonically into linear and non-linear parts:

$$0 \to \pi_q(A) \oplus \pi_q(B) \to \pi_q(A \vee B) \to \pi_q(A \vee B) \to 0,$$

where $\pi_q$ is the cokernel, one may obtain a canonical decomposition of the $E^1$ term of an ordered covering into linear and nonlinear parts (cf. §5)

$$0 \to \oplus E_{p,q}^1 \to E_{*,q}^1 \to \star E_{*,q}^1 \to 0.$$

This is an exact sequence of simplicial groups, and so passing to homotopy yields a long exact sequence

$$\cdots \to \oplus E_{p,q}^2 \to E_{*,q}^2 \to \star E_{*,q}^2 \to \cdots.$$

The linear part $\oplus E_{p,q}^2$ can be calculated by means of non-degenerate intersections (cf. §V) and therefore handles comparatively easily. On the other hand, the non-linear part vanishes for $q < 2k - 1$ in the case of a $k$-connected covering.

The method of proof of the theorem is as follows: Let $G_*$ be a bisimplicial group (cf. §VII). We construct a pair of spectral sequences $\mathcal{E}_{p,q}^*$, $\mathcal{E}_{p,*}^*$ with common convergence such that

$$\mathcal{E}_{p,q}^2 = \pi_p^h(\pi_q^v(G)) \quad \text{and} \quad \mathcal{E}_{p,*}^2 = \pi_p^v(\pi_q^h(G)),$$

where $\pi_q^v(G)$ is the simplicial group obtained by taking the $q$-th homotopy groups of $G_p$, for each $p$, and where $\pi_p^h(\pi_q^v(G))$ is the $p$-th homotopy group of this simplicial group ($v = \text{vertical}$, $h = \text{horizontal}$).

Our original proof produced spectral sequences converging to the homotopy groups of a certain “total simplicial group” $\mathcal{G}$ associated to $G_*$. Quillen subsequently simplified things greatly by obtaining similar spectral sequences converging to the homotopy groups of the “diagonal” simplicial groups $\Delta G$ associated to $G_*$ [10], rendering publication of our proofs for $\mathcal{G}$, superfluous. We have, however, included the definition of $\mathcal{G}$ in §3.
To apply this pair of spectral sequences, suppose \( f: U \to X \) a surjective morphism of reduced simplicial sets. Letting \( G \) denote Kan’s functor [5], form the double simplicial group \( G_p = G(U^{p+1}) \), where \( U^{p+1} \) is the fibre product as above. Consider the second spectral sequence \( *E_{p,q} \) for \( G_p \). We show in §IV that \( *E_{p,q}^2 \) is degenerate in the sense that \( *E_{p,q}^2 = 0 \) if \( q > 0 \) and \( *E_{p,0}^2 \approx \pi_p(X) \). It follows that \( \pi_q(X) \) is isomorphic to the abutment of the two spectral sequences, and so the first spectral sequence yields the theorem.

We have profited from stimulating discussions with E. Curtis and D. Kan.

### §II. Examples and Calculations

It is clear that the necessary fuel for the spectral sequence of a covering is knowledge of the behavior of the homotopy groups of wedges. Fortunately there is the Hilton-Milnor theorem [3, 8, 9] which gives rather precise information of this kind. To shed light on the nature of the homotopy spectral sequence of a morphism we will consider a few interesting examples, whose \( E^2 \) terms we will compute up to and including the first few dimensions in which nonlinear terms occur.

1. **The van Kampen theorem**

   Let us investigate the first non-zero term in the case of an ordered covering of a reduced simplicial set \( X \), where the covering is by two subsets \( U \) and \( V \), and \( U \cap V = W \). Then

   \[
   E^1_{0,1} \xleftarrow{{d_0}} E^1_{1,1}
   \]

   is just

   \[
   \pi_1(U) \ast \pi_1(V) \leftarrow \pi_1(U) \ast \pi_1(W) \ast \pi_1(V)
   \]

   where \( d_1 \), \( d^0 \) restricted to \( \pi_1(U) \) and \( \pi_1(V) \) are the identity maps, and restricted to \( \pi_1(W) \) are induced from the inclusions of \( W \) in \( U \) and \( V \) respectively. It follows that \( E_{0,1} = \pi_0(E_{1,1}) \) (which is the cokernel of \( (d^0, d^1) \) in the category of groups) is the free product of \( \pi_1(U) \) and \( \pi_1(V) \) amalgamated over \( \pi_1(W) \).

2. **The generalized Freudenthal suspension theorem**

   Suppose we are given an ordered covering \( U \vee V \to Y \), where \( U \) and \( V \) are contractible subcomplexes of \( Y \) and where \( U \cap V = X \). Then \( Y \) is of the homotopy type of the suspension \( SX \) of \( X \), and the spectral sequence reads

   \[
   E^1_{pq} = \pi_q \left( \vee^p X \right) \Rightarrow \pi_{p+q}(SX),
   \]

   where \( \vee^p X \) is the \( p \)-fold wedge. Say that \( X \) is \( (k - 1) \)-connected, with \( k \geq 2 \). Since the linear part of the spectral sequence may be computed from non-degenerate intersections (§5), it is immediate that

   \[
   \rho E^1_{pq} = \pi_q(X) \quad \text{and} \quad \rho E^2_{pq} = 0 \quad \text{if} \quad p \neq 1.
   \]
It follows from [3] that \( \#E_{pq}^1 = 0 \) and hence \( \#E_{pq}^2 = 0 \) if \( q < 2k - 1 \). Also \( \#E_{pq}^1 = \#E_{pq}^2 = 0 \) for \( p = 0, 1 \). The critical dimension is \( q = 2k - 1 \), and by [3] we have

\[
\#E_{p,2k}^1 = \#\pi_{2k-1} \left( \bigvee X \right) = \bigoplus_{i \neq p} \left( \pi_a(X) \otimes \pi_b(X) \right).
\]

A simple calculation shows that therefore

\[
\#E_{2,2k-1}^2 = \#E_{2,2k-1}^1 = \pi_a(X) \otimes \pi_b(X),
\]

and

\[
\#E_{3,2k-1}^2 = 0.
\]

This yields the classical Freudenthal suspension theorem, together with the exact sequence

\[
\begin{align*}
\pi_{2k}(X \vee X) & \longrightarrow \pi_{2k}(X) \otimes \pi_{2k+1}(SX) \longrightarrow \pi_{2k}(X) \otimes \pi_{2k+1}(X) \\
\sigma & \longrightarrow \pi_{2k-1}(X) \otimes \pi_{2k}(SX) \longrightarrow 0.
\end{align*}
\]

where \( \eta \) is the alternating sum of the homomorphisms induced from the maps \( 1 \vee 0, 1 \vee 1, 0 \vee 1 \) of \( X \vee X \) to \( X \). If one wishes to carry this computation to higher dimensions, one must determine for example \( \#E_{p,2k-1}^2 \) for \( p > 3 \). This leads to questions regarding the homotopy groups of \( L^2 K(\pi, 1) \) where \( L^2 \) is Kan's functor on free abelian group complexes [6].

3. The suspension of non-simply connected spaces

The suspension spectral sequence provides striking information even in the extreme case of a non-simply-connected space \( X \). However, the appearance of the spectral sequence is quite different, as can be shown by the following examples: Let \( X = S^n \), first where \( n > 1 \). Then the \( E^2 \) term is as elucidated above. For \( n = 1 \), however, the \( E^1 \) term is a free simplicial group concentrated on the horizontal line \( q = 1 \).

More generally, let \( X \) be any \( K(G, 1) \) space. Then the successive wedges \( \bigvee X \) are again \( K(G, 1) \) spaces, and the \( E^1 \) term is a non-abelian simplicial group concentrated on the horizontal line \( q = 1 \). Call this simplicial group \( *(G) \). Thus \( *(G)_n \) is the free product of \( n \) copies of the group \( G \). From the spectral sequence, we may conclude that the homotopy groups of \( *(G) \) are equal to those of \( SK(G, 1) \) with a shift in dimension by 1.

Now take any simplicial set \( X \) whose fundamental group is \( G \). Form the suspension spectral sequence for \( X \). It is clear that the simplicial group \( E_{*1}^1 \) is \( *(G) \). Therefore, \( E_{p,1}^1 = \pi_{p+1}(SK(G, 1)) \). Consider the group \( E_{12}^2 \) which is the cokernel of the map \( \eta: \pi_2(X \vee X) \rightarrow \pi_2(X) \) defined above. It is not difficult to show that this cokernel is just the quotient \( \pi_2(X) \) of \( \pi_2(X) \) under the action of \( \pi_1(X) \) (we omit the proof). Thus the spectral sequence yields the exact sequence

\[
\begin{align*}
\pi_4(SK(G, 1)) & \longrightarrow \pi_5(X) \otimes \pi_3(SX) \longrightarrow \pi_5(SX) \longrightarrow \pi_5(SK(G, 1)) \longrightarrow 0.
\end{align*}
\]
4. The operation of attaching a cell

Given a topological space $X$ and a continuous map $\varepsilon : S^n \to X$, we may form the space $Y = X \cup D^{n+1}$, where $D^{n+1}$ is the $n + 1$ cell. The question is: what is the relationship between the homotopy groups of $X$ and those of $Y$? If $X$ is $(k - 1)$-connected, the groups $\pi_q(Y)$ have been computed up to dimension $k + n$ by Massey [7] using the homotopy theory of triads.

We translate the question into the context of reduced simplicial sets as follows: Let $X \subset Y$ be reduced simplicial sets, and $\Delta^{n+1} \subset Y$ a subsimplicial set isomorphic to the $n + 1$ simplex, such that $Y = X \cup \Delta^{n+1}$ and $X \cap \Delta^{n+1} = \partial \Delta^{n+1}$. Denote $\partial \Delta^{n+1} = S^n$. Then the ordered covering

$$f : X \vee \Delta^{n+1} \to Y$$

gives rise to a spectral sequence

$$E^1_{pq} = \pi_q\left(X \vee \vee S^n\right) \Rightarrow \pi_{p+q}(Y).$$

The linear part of this spectral sequence is immediately seen to be $E^2_{pq} = 0$ if $p > 1$, while $E^2_{pq}, E^2_{0q}$ are the kernel and cokernel respectively of the map $\alpha_q : \pi_q(S^n) \to \pi_q(X)$ induced by the inclusion.

Suppose that $X$ is homotopic to the suspension of a $(k - 2)$-connected space, and $n > k > 1$. (Much of what we say is true more generally for $(k - 1)$-connected spaces.) By the Hilton-Milnor theorem [8] combined with the Freudenthal suspension theorem, we have

$$#\pi_q(X \vee S^n) \approx \pi_q(S^{n-1}X) \approx \pi_{q-n+1}(X)$$

and the $p$ natural inclusions

$$X \vee S^n \to X \vee \vee S^n$$

induce an isomorphism

$$\bigoplus_p #\pi_q(X \vee S^n) \cong #\pi_q\left(X \vee \vee S^n\right)$$

if $k + n - 1 \leq q < \min(2n - 1, 2k + n - 2)$. Thus we may compute:

$$#E^2_{1q} = \pi_{q-n+1}(X) \quad \text{and} \quad #E^2_{pq} = 0 \quad \text{if} \quad p \neq 1$$

for $k + n - 1 \leq q < \min(2n - 1, 2k + n - 2)$.

We deduce for this range of $q$ that

$$E^2_{pq} = 0 \quad \text{if} \quad p > 1$$

and obtain an exact sequence

$$0 \to \ker \alpha_q \to E^2_{1q} \to \pi_{q-n+1}(X) \to \cok \alpha_q \to E^2_{0q} \to 0.$$

Thus we may compute $\pi_q(Y)$ explicitly in this range.

In general, we may of course express $E^2_{pq}$ as the cokernel of a homomorphism $\eta : \pi_q(X \vee S^n) \to \pi_q(X)$. The homomorphism $\eta$ is just the boundary operator of the complex $E^2_{pq}$ in degree 1. Explicitly $\eta = \eta_1 - \eta_2$ with $\eta_1, \eta_2$ the induced maps on homotopy
coming from the maps $1 \vee 0$ and $1 \vee \alpha : X \vee S^n \to X$. In terms of $\eta$ we can calculate $\pi_q(Y)$ for one higher dimension, $q \leq \min(2n-1, 2k+n-2)$ by the exact sequence

$$\pi_q(X \vee S^n) \longrightarrow \pi_q(X) \longrightarrow \pi_q(Y) \longrightarrow E^2_{1,q-1} \longrightarrow 0.$$ 

In the next range of dimensions, one must take into account the further cross-terms, e.g. when $q = 2n-1 < 2k+n-2$ we have:

$$E^2_{1,2n-1} = \pi_q(X)/(\alpha).$$

5. The action of a finite group

Take the simplest case that might arise in our setting. Suppose $G$ is a finite group acting on a reduced simplicial set $X$ so as to act freely on all non-degenerate simplices of positive degree. (Since the base point is unique, the action has no choice but to preserve it.) Let $f : X \to W$ denote the natural map to the orbit-space $W$ of the action. We shall investigate the spectral sequence of the morphism $f$. Note that in this case, the fibre powers of $X$ over $W$ are again iterated wedges of $X$. In fact, $E^1_{pq} = \pi_q(\vee X)$, where $n$ denotes the number of elements of the group $G$.

Thus, in a manner analogous to the case of a covering, we may consider the “linear part” of $E^1_{aq}$. That is, we form the canonical subsimplicial group given by: $\oplus E^1_{pq} = \oplus \pi_q(X)$.

Since the complex $\oplus E^1_{aq}$ is seen to be the canonical resolution of the $G$-module, $\pi_q(X)$, we get: $\oplus E^2_{pq} = H_p(G, \pi_q(X))$, and as usual, for $X$ $(k-1)$-connected, we have $\oplus E^2_{pq} = 0$ if $q < 2k-1$.

§III. THE TOTAL SIMPLICIAL SET ASSOCIATED TO A BISIMPLICIAL SET

We will use terminology similar to that of [2]. In particular, we use the term simplicial rather than semi-simplicial [5] in order to avoid the unfortunate word bisemi-simplicial. Thus $\Delta$ denotes the category of “simplex types” whose objects are $\Delta_p = \{0, \ldots, p\}$ for $p = 0, \ldots$ and whose maps are the (weakly) increasing maps $\Delta^i_p \to \Delta^j_q$. A simplicial object $X$ with values in a category $C$ is a (contravariant) functor $X : \Delta^\circ \to C$.

Consider the product $\Delta \times \Delta$ whose objects are pairs $(\Delta_p, \Delta_q)$ of simplex types and whose maps are pairs of weakly increasing maps. A (contravariant) functor $X : (\Delta \times \Delta)^\circ \to C$ is called a bisimplicial object with values in $C$. To give $X$ is equivalent with giving for each $(p, q)$ an object $X_{p,q}$ of $C$ and morphisms

$$d^i : X_{p,q} \to X_{p-i,q}$$
$$s^i : X_{p,q} \to X_{p+1,q} \quad i = 0, \ldots, p$$
$$\delta^i : X_{p,q} \to X_{p,q-1}$$
$$\sigma^j : X_{p,q} \to X_{p,q+1} \quad j = 0, \ldots, q$$

such that the maps $d^i, s^i$ commute with $\delta^i, \sigma^j$ and that $d^i, s^i$ (resp. $\delta^i, \sigma^j$) satisfy the standard identities [5]. We think of $d^i, s^i$ as the “horizontal” operators and of $\delta^i, \sigma^j$ as the “vertical” operators.
A (bi)simplicial object with values in the category of sets (resp. groups) is called a (bi)simplicial set (resp. group). If $X$ is a bisimplicial set, it is convenient to think of an element of $X_{p,q}$ as a product of a $p$-simplex and a $q$-simplex.

We are going to describe a functor $T$ from bisimplicial objects to simplicial objects. In order not to make the exposition too heavy, we assume that we are dealing with bisimplicial sets. However, it is clear that the construction by means of finite inverse limits is categorical and hence yields a functor whenever the category $C$ of values is closed under these limits.

Let $X$ be a bisimplicial set. We call the simplicial set which we construct the total simplicial set $T(X) = T$ associated to $X$. Unfortunately $T(X)$ need not satisfy the Kan condition even if $X$ satisfies it for both gradations. However, $T(X)$ is naturally a simplicial group if $X$ is a bisimplicial group, and so the Kan condition is satisfied in that case.

Put

$$X_{(n)} = \prod_{p+q=n} X_{p,q},$$

and define $T_n \subset X_{(n)}$ as follows: An element $(x_0, \ldots, x_n)$ of $X_{(n)}$, with $x_p \in X_{p,n-p}$ is in $T_n$ iff

$$\delta^0 x_p = d^{p+1} x_{p+1} \quad \text{for} \quad p = 0, \ldots, n - 1$$

(we are suppressing the second index whenever possible). Next, define the faces and degeneracies

$$D^j : T_n \to T_{n-1} \quad \text{for} \quad j = 0, \ldots, n$$

by

$$D^j(x) = (\delta^j x_0, \delta^{j-1} x_1, \ldots, \delta^1 x_{j-1}, d^j x_j, d^j x_{j+1}, \ldots, d^j x_n)$$

$$S^j(x) = (\sigma^j x_0, \sigma^{j-1} x_1, \ldots, \sigma^0 x_j, s^j x_j, s^j x_{j+1}, \ldots, s^j x_n).$$

We leave it to the reader to verify that $T(X) = \{T_n : D^j, S^j\}$ is a simplicial set. It is clear that if $X$ is a bisimplicial group then $T_n$ is a group and $D^j, S^j$ are group homomorphisms, hence $T(X)$ is a simplicial group.

It is useful for one's intuition to interpret an element $x \in T_n$ as a union of products of simplexes $x = x_0 \cup \cdots \cup x_n$ (where $x_p$ is a product of a $p$-simplex and a $n-p$-simplex) having certain faces in common. For $n = 2$, $x$ is depicted below. We have drawn the "vertical" unit length longer than the "horizontal" unit length. Note the faces and degeneracies of the figure.
In the figure, the labelling \((a, b)\) of the vertices indicates the first bisimplex, \(x_a\), and the last one, \(x_b\), to which the vertex belongs. The vertices \((0,0)\), \((1,1)\), \((2,2)\) are the vertices of \(x\), and each bisimplex has its orientations induced by the labelling of its vertices.

The relationship between the homotopy of a bisimplicial group and its associated total simplicial group is given by the following:

**Theorem.** Let \(G\) be a bisimplicial group and \(\bar{G}\) its associated total simplicial group. There is a pair of spectral sequences of homological type

\[
\begin{align*}
\ E^2_{p,q} &= \pi_p^h(\pi_q^c(G)) \Rightarrow \pi_{p+q}(\bar{G}) \\
\ E^2_{p,q} &= \pi_p^v(\pi_q^c(G)) \Rightarrow \pi_{p+q}(\bar{G}).
\end{align*}
\]

Since the spectral sequence constructed by Quillen [10] works equally well for our application, we omit the proof.

**§IV. SIMPLICIAL RESOLUTIONS**

Let \(X\) be an object of a category \(C\) and \(K\) a simplicial object of \(C\). An augmentation \(\varepsilon\) of \(K\) into \(X\) is a map \(K_0 \to X\) such that the maps \(\varepsilon_0, \varepsilon_1 : K_1 \to X\) are equal. Suppose \(C\) is the category of sets. We call \(\varepsilon\) a simplicial resolution of \(X\) if the induced map from the geometric realization of \(K\) to the discrete space \(X\) is a homotopy equivalence, i.e. if \(\pi_0(K) \approx X\) and if each connected component of \(K\) is contractible. The following lemma is an immediate consequence of [8]:

**Lemma.** Let \(\varepsilon : K \to X\) be a simplicial resolution of a pointed set \(X\). Let \(FK\) denote the free simplicial group on \(K\) and \(FX\) the free group on the pointed set \(X\) (where the base points of \(K\) (resp. \(X\)) are set equal to 1). Then the canonical map \(FK \to FX\) is a simplicial resolution.

In fact, it is shown in [8] that the homotopy type of \(FK\) is that of the loop space of \(SK\), which is homotopic to the suspension \(SX\) of the discrete space \(X\), and \(SX\) is a \(K(FX,1)\).

Now let \(X\) be a simplicial set and \(K\) a bisimplicial set. View \(K\) as a “horizontal” simplicial object with values in the “vertical” simplicial sets \(K_\rho\) \((\rho = 0, \ldots)\). An augmentation \(\varepsilon : K_0 \to X\) will be called a simplicial resolution of \(X\). if for each \(\rho \geq 0\) the augmentation of the simplicial set \(K_\rho\) into \(X_\rho\) given by \(\varepsilon_\rho : K_\rho \to X_\rho\) is a simplicial resolution of \(X_\rho\).

Suppose we are given such a simplicial resolution where in addition \(X\) is a reduced complex and \(K_0\) consists of one point for each \(p \geq 0\). Denote by \(G(X)\) the simplicial group of Kan [5] and by \(G^n(K)\) the bisimplicial group obtained by applying the functor \(G\) to each “vertical” simplicial set \(K_\rho\), viz.

\[
(G^n(K))_\rho = (G(K_\rho),)_\rho.
\]

Since \(G\) is a functor this is a bisimplicial group, and induces an augmentation of \(G^n(K)\) to \(G(X)\).

**Proposition.** With the above notation, the augmentation of \(G^n(K)\) into \(G(X)\) is a simplicial resolution.
Proof. If $Y$ is any reduced simplicial set then by definition [5] of $G$, the group $(GY)_n$ is the free group on the set $Y_{n+1}$, with the relations $\sigma^n y = 1$ for every $y \in Y_n$. Thus $(GY)_n$ is the free group on the pointed set obtained from $Y_{n+1}$ by contracting the subset $\sigma^n Y_n$ to a point.

Consider the following diagram, in which the vertical maps $\sigma^n$ are inclusions:

$$
\begin{array}{c}
X_{n+1} & \leftarrow & K_{0,n+1} & \leftarrow & K_{1,n+1} & \cdots \\
\uparrow \sigma^n & & \uparrow \sigma^n & & \uparrow \sigma^n \\
X_n & \leftarrow & K_{0,n} & \leftarrow & K_{1,n} & \cdots 
\end{array}
$$

(*)

Let

$$
\begin{array}{c}
X_{n+1} & \leftarrow & K_{0,n+1} & \leftarrow & K_{1,n+1} & \cdots 
\end{array}
$$

(**)

be the diagram obtained by contracting the image of $\sigma^n$ to a point. Then the augmented resolution of $(GX)_q$ by $(G^rK)_q$ is just the free group on the pointed diagram (**). Since we are to show that this is a simplicial resolution of $(GX)_q$, we are reduced by the lemma to showing that (** is a simplicial resolution of $X_{n+1}$. But this is immediately seen since, by assumption, the horizontal lines of (*) are simplicial resolutions of $X_{n+1}$ and $X_n$ respectively.

Let us apply the theorem of §III or the spectral sequence of [10] to the bisimplicial group $G^r(K)$ in the above context. It follows from the proposition that

$$
\pi_q^r(G^rK) = \begin{cases} 
GX & \text{if } q = 0 \\
0 & \text{if } q > 0.
\end{cases}
$$

Hence we have

$$
\pi^p_q(GX) = \pi_{p+1}(X) \quad \text{if} \quad q = 0
$$

and so the spectral sequence gives

$$
\pi^p_q(GX) \approx \pi_q^r(G^rK)
$$

where $G^rK$ is the total simplicial group (or the diagonal simplicial group if one is in the context of [10] associated to $G^rK$. Thus the first spectral sequence abuts to the associated graded group of a certain filtration of $\pi_p(GX)$. Shifting the $p$-dimension by 1, we obtain

COROLLARY. Let $K_.$ be a simplicial resolution of the reduced simplicial set $X$. and suppose that $K_{p0}$ consists of one point for all $p$. There is a spectral sequence of homological type

$$
E^2_{p,q} = \pi_p^r(K.) \Rightarrow \pi_{p+q}(X).
$$

In order to obtain the theorem of §I, let $K_.$ be the bisimplicial set

$$
K_{p,q} = (U^{p+1})_{q}
$$

obtained from the surjective map $f: U \rightarrow X$ of reduced simplicial sets. The map $f: U = K_0 \rightarrow X$ is clearly an augmentation of $K_.$ into $X$. Since $U.$ is reduced, so is each $U^t$, and hence $K_{p0}$ consists of one point for each $p$. Thus it remains to show that the augmentation is a simplicial resolution of $X$. Let $x \in X_q$, and let $\{u_a\}$ be the (non-empty) set of simplexes of $U_q$ with $f(u_a) = x$. Then the simplexes of $(U^{p+1})_{q} = K_{pq}$ lying over $x$ are just $(p+1)$-tuples
(u_0, ..., u_p) of u_q's. Thus the elements of K_q lying over x form a formal (unoriented) simplex with vertices \{u_a\}, which is a contractible simplicial set. This completes the proof of the theorem. A similar argument yields the spectral sequence of an ordered covering—the elements of K_q lying over x form in that case an oriented simplex.

§V. THE LINEAR PART

Recall that if A = \bigvee_{j=1} A_j is a wedge of spaces, canonical inclusions and retractions:

\[
A_j \xrightarrow{i_j} A \xleftarrow{r_j} A_j
\]
yield a canonical split exact sequence for q > 1:

\[
0 \to \bigoplus_j \pi_q(A_j) \xrightarrow{\pi_q(A)} \pi_q(A) \to 0
\]

where \(\pi_q(A)\) is defined to be the cokernel.

Suppose now that \(\varepsilon: \bigvee U_j \to X\) is an ordered covering. Denote by \(J^{p+1}\) the set of vectors \((j_0, ..., j_p)\) with \(j_0 < j_1 \leq \cdots \leq j_p\). Put

\[
\oplus E^1_{p,q} = \bigoplus_{(j) \in J^{p+1}} \pi_q(U_{(j)})
\]

and

\[
\oplus E^1_{p,q} = \pi_q \left( \bigvee_{(j) \in J^{p+1}} U_{(j)} \right).
\]

We may endow \(\oplus E^1_{*,q}\) and \(\oplus E^1_{*,q}\) with face and degeneracy operators, and obtain a canonical exact sequence of simplicial groups:

\[
0 \to \oplus E^1_{*,q} \to E^1_{*,q} \to \oplus E^1_{*,q} \to 0.
\]

The splitting does not conserve the simplicial group structures and so the above exact sequence yields upon passage to homotopy, an exact sequence:

\[
... \to \oplus E^2_{p,q} \to E^2_{p,q} \to \oplus E^2_{p,q} \to \oplus E^2_{p-1,q} \to ...
\]

The group \(\oplus E^2_{p,q}\) is called the linear part. Suppose that the covering \(\varepsilon\) is simply connected (\(\varepsilon\) is called k-connected if \(U_{(j)}\) is k-connected for all \((j)\)). Then \(\oplus E^1_{*,q}\) is an abelian simplicial group for all \(q \geq 0\), and its homotopy may be computed as the homology of the abelian complex of groups \(\{\oplus E^1_{*,q}; d\}\) where \(d = \sum_{i=0}^p (-1)^i \cdot \oplus E^1_{p-i,q} \to \oplus E^1_{p-1,q}\). Here is the great simplification that occurs in computation of the linear part of \(E^2: \oplus E^2_{p,q}\) may be computed by considering only non-degenerate intersections \(U_{(j)}\). To be more explicit, define \(J^i \subset J\) to be the set of vectors \((j)\) such that \(j_0 < j_1 < \cdots < j_i\), and consider:

\[
\oplus E^1_{p,q} \subset \oplus E^1_{p,q}
\]

defined by:

\[
\oplus E^1_{p,q} = \bigoplus_{(j) \in J^{p+1}} \pi_q(U_{(j)})
\]

(the non-degenerate part of \(\oplus E^1_{p,q}\)). \(\oplus E^1_{p,q}\) is a subcomplex with differential operator \(d\).
PROPOSITION. (4.1) The inclusion of abelian complexes,
\[ \vartheta_{E^1_* \rightarrow E^1_*} \]
induces an isomorphism on homology.

Proof. This is standard, and reflects the following more precise fact (see e.g. §17 of [6]): There is a functor (normalization) \( N \) from abelian group complexes to chain complexes which are zero in negative dimensions. This functor \( N \) commutes with homology functors and establishes an equivalence between the two categories. One then checks that the above inclusion gives an isomorphism between \( \vartheta E^1_* \) and \( N(\vartheta E^1_* ) \). Using the well-known fact that \( \pi_\ast(A \vee B) = 0 \) if \( A, B \) are \( n \)-connected, \( m \)-connected, respectively, and \( q < n + m - 1 \), we have for a \( k \)-connected covering \( E \), \( \vartheta E^1_\ast = 0 \) \( q < 2k - 1 \).

REFERENCES
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Harvard University