Note

The Fundamental Theorem of Algebra: An Elementary and Direct Proof

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Here is a simple, differentiation-free, integration-free, trigonometry-free, direct, and elementary proof of the Fundamental Theorem of Algebra.

One can ask, as soon as the complex numbers have been defined, whether every polynomial has a zero in the complex numbers. In this note I consider how early in the development of complex analysis this question can be answered.

As pointed out by Remmert [11], Burckel [2], and others, two of the best analytical proofs of the FTA are the easy and short but not elementary one given by Argand [1] (see also [3, 4, 6, 7, 10, 11, 12, 14]), and the elementary but not so easy or short one given by Littlewood [9] (see also [5, 8, 11, 13]). All these works, except [2], use or prove d'Alembert's Lemma [14] or Argand's Inequality [11]:

> "If P is a nonconstant complex polynomial and $P(z_0) \neq 0$, $z_0 \in \mathbb{C}$, then any neighborhood of z_0 contains a point w such that $|P(w)| < |P(z_0)|$".

The proof of the FTA I will now present does not apply d'Alembert's Lemma or Argand's Inequality. Instead I assume without proof only the continuity of complex polynomials and the following consequences of the completeness of $\mathbb{R}$:

- Any continuous function $f : D \to \mathbb{R}$, D a bounded and closed disc, has a minimum on D.
- Every positive real number has a positive square root.

Square Roots. It is well known that $z^2 = a + ib$, $a, b \in \mathbb{R}$, is solvable in $\mathbb{C}$. We have $\pm z = \sqrt{\frac{a}{2} + \frac{\sqrt{a^2+b^2}}{2}} + i\operatorname{sgn}(b)\sqrt{-\frac{a}{2} + \frac{\sqrt{a^2+b^2}}{2}}$, with $\operatorname{sgn}(b) = \frac{b}{|b|}$, if $b \neq 0$, and $\operatorname{sgn}(0) = 1$. Applying this formula repeatedly we find all the 2^j-roots, $j \in \mathbb{N}$, of $z = \pm 1$ and $z = \pm i$.

Fundamental Theorem of Algebra. *Let P be a complex polynomial, with degree$(P) = n \geq 1$. Then there exists $z_0 \in \mathbb{C}$ satisfying $P(z_0) = 0$.*

Proof. Writing $P(z) = a_0 + a_1 z ... + a_n z^n$, with $a_j \in \mathbb{C}$, $0 \leq j \leq n$, $a_n \neq 0$, we have

$$|P(z)| \geq |a_n||z|^n - |a_0| - ... - |a_{n-1}||z|^{n-1},$$

from which follows $\lim_{|z| \to \infty} |P(z)| = \infty$. By continuity, it is easy to see that $|P|$ has an absolute minimum at some $z_0 \in \mathbb{C}$. Suppose without loss of generality that $z_0 = 0$. Hence, putting $S^1 = \{\omega \in \mathbb{C} : |\omega| = 1\}$,

$$|P(r\omega)|^2 - |P(0)|^2 \geq 0, \quad \forall r \geq 0, \quad \forall \omega \in S^1, \qquad (1)$$

and $P(z) = P(0) + z^k Q(z)$, for some $k \in \{1, ..., n\}$, where Q is a polynomial and $Q(0) \neq 0$. Substituting this expression,

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at $z = r\omega$, in inequality (1), we get

$$|P(0) + r^k\omega^k Q(r\omega)|^2 - |P(0)|^2 = 2r^k \mathrm{Re}[\overline{P(0)}\omega^k Q(r\omega)] + r^{2k}|Q(r\omega)|^2 \geq 0, \quad \forall r \geq 0, \quad \forall \omega \in S^1,$$

and, dividing by $r^k > 0$,

$$2\mathrm{Re}\left[\overline{P(0)}\omega^k Q(r\omega)\right] + r^k|Q(r\omega)|^2 \geq 0, \quad \forall r > 0, \quad \forall \omega \in S^1,$$

whose left side is a continuous function of r, $r \in [0, +\infty)$. Thus, letting $r \to 0$, we have

$$2\mathrm{Re}[\overline{P(0)}Q(0)\omega^k] \geq 0, \quad \forall \omega \in S^1. \qquad (2)$$

Let $\alpha = \overline{P(0)}Q(0)$. Factoring out powers of 2, we can write $k = 2^j m$ where m is odd. Taking $\omega = 1$ in (2) we conclude that $\mathrm{Re}[\alpha] \geq 0$. Choosing ω so that $\omega^{2^j} = -1$, and thus $\omega^k = -1$, we conclude that $\mathrm{Re}[\alpha] \leq 0$. Hence $\mathrm{Re}[\alpha] = 0$. Choosing ω so that $\omega^{2^j} = i$, we conclude that $\omega^k = \pm i$ and $\overline{\omega}^k = \mp i$. Substituting ω and $\overline{\omega}$ into (2), we conclude that $\mathrm{Im}[\alpha] = 0$. So $\alpha = 0$, and $P(0) = 0$.

Remark The last paragraph of the proof was a trick to avoid appealing to trigonometry. The proof that (2) implies $P(0) = 0$ can be easily done with the help of De Moivre's Formula $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$, n natural and θ real, as follows. Putting $\omega = \cos\theta + i\sin\theta$, $\omega \in S^1$ and θ a real number, we choose values of θ so that $\omega^k = \cos k\theta + i\sin k\theta$, k as in the above proof of the FTA, assumes the values ± 1 and $\pm i$. Hence we get $\mathrm{Re}[\pm\overline{P(0)}Q(0)] \geq 0$ and $\mathrm{Re}[\pm\overline{P(0)}Q(0)i] \geq 0$, and conclude that $P(0) = 0$.

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